Lecture 15: Matrix Multiplication

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We know $M_{mn}(\mathbb{R})$ is a vector space. ie. we can add any 2 mxn matrices and we can multiply any mxn matrix by a scalar.
Here is another operation:

$$\begin{array}{c} m \end{array}\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{p} \quad \begin{array}{c} p \end{array}\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \\ 0 & 7 & -2 \end{bmatrix}}_{m} = \begin{array}{c} m \end{array}\underbrace{\begin{bmatrix} -1 & 25 & -1 \\ -1 & 52 & 5 \end{bmatrix}}_{n}$$

<u>Always remember</u>: row x column

<u>14.1 How to multiply a matrix by another matrix</u>

<u>Example</u>

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \times \underset{B}{\begin{bmatrix} 5 & 2 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

to multiply two matrices, the # of columns in A must equal the # of rows in B

- dimension of product matrix = **# of rows in A by # of columns in B**
  i.e. in this example, A has 2 rows and B has 2 columns, so the product matrix is 2 rows by 2 columns
- each element of the product matrix has a unique address; [1,1], [1,2], [2,1], [2,2] (row, column)
- element [1,1] is equal to the dot product of the first row of A and the 1st column of B (=19)
  - every element [x,y] of the product matrix is equal to $\underbrace{A_x}_{\text{row } x \text{ in } A} \cdot \underbrace{B_y}_{\text{column } y \text{ in } B}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 5(1)+2(1)+3(4) & 2(1)+2(3)+3(2) \\ 4(5)+5(1)+6(4) & 4(2)+5(3)+6(2) \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 14 \\ 49 & 35 \end{bmatrix}$$

<u>More Examples</u>

a) $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

b) $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 3 \\ 2 & 7 \end{bmatrix}$

Each column is a linear combination of the columns of A

For a matrix A and a matrix $B = [b_1|b_2|\ldots|b_n]$ we can say:

$$A \cdot B = [Ab_1|Ab_2|\ldots|Ab_n]$$

c) <u>Relation to linear systems</u>

<u>Problem</u>: Try to solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

vector equation $\quad x_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

linear system
$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6 \end{aligned}$$

<u>Solution</u>:

$$x_1 + 2x_2 = 5$$
$$3x_1 + 4x_2 = 6$$

Solution:
$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -2 & -9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 9/2 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 9/2 \end{array}\right] \quad S = \left\{ \begin{pmatrix} -4 \\ 9/2 \end{pmatrix} \right\}$$
$$x_1 = -4$$
$$x_2 = \frac{9}{2}$$

d) $$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 5 & 0 \end{bmatrix}$$

14.2 Strange properties of multiplying matrices

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

$A \cdot B = \begin{bmatrix} 5 & 6 & -1 \\ 11 & 12 & -3 \end{bmatrix}$ ← -1 1st col of A, 0 2nd col of A

$B \cdot A = not\ defined!$

b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A \cdot B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, B \cdot A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

ie. matrix multiplication is *non-commutative*

c) $A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ $\qquad n$ x 1 matrices (column vectors)

$A^T B = 8 \quad (= A \cdot B - dot\ product\ of\ vectors)$
$B^T A = 8 \quad (= B \cdot A - dot\ product\ of\ vectors)$ } commutative

$$AB^T = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot [4 \ -1 \ \ 2] = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 0 & 0 \\ 8 & -2 & 4 \end{bmatrix}$$
$$BA^T = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \cdot [1 \ 0 \ 2] = \begin{bmatrix} 4 & 0 & 8 \\ -1 & 0 & -2 \\ 2 & 0 & 4 \end{bmatrix}$$

(=tensor products of vectors) ← non-commutative

d) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{2x2}$

Note: we can read $A \cdot B$ as follows:

The first column of the product is $1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the second one is $2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

e) $A = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 7 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}, BC = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$

14.3 Pleasant properties

NOTE: Notation:

$$0_{mxn} = m \updownarrow \underbrace{\begin{bmatrix} 0 & & 0 \\ & & \\ 0 & & 0 \end{bmatrix}}_{\longleftarrow n \longrightarrow}$$

"zero matrix"

$$I_m = m \updownarrow \underbrace{\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & & \\ 0 & 0 & & 1 \end{bmatrix}}_{\longleftarrow m \longrightarrow}$$

"identity matrix"

Theorem

Let A, B, C be matrices and $k \in \mathbb{R}$ be a scalar. Then, whenever defined:

a) $(AB)C = A(BC)$
b) $A(B + C) = AB + AC$
c) $(B + C)A = BA + CA$
d) $k(AB) = (kA)B = A(kB)$
e) $(AB)^T = B^T A^T$
f) If A is m x n, then $I_m A = A$ and $AI_n = A$
g) If A is m x n, then $A \cdot 0_{nxp} = 0_{mxp}$